PHYS 705: Classical Mechanics

Constraints and Generalized Coordinates

Constraints

Problem Statement:

In solving mechanical problems, we start with the 2nd law

$$\sum_j \mathbf{F}_{ji} + \mathbf{F}_i^{(e)} = m_i \ddot{\mathbf{r}}_i \qquad (*)$$

In principle, one can solve for $\boldsymbol{r}_i(t)$ (trajectory) for the i^{th} particle by specifying all the external and internal forces acting on it .

However, if constraints are present, these external forces in general are NOT known.

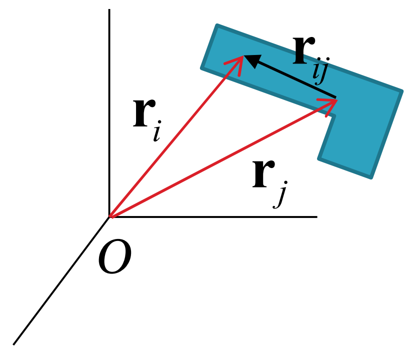
Therefore, we need to understand the various constraints and know how to handle them.

Holonomic Constraints

Holonomic constraints can be expressed as a function in terms of the coordinates and time,

$$f\left(\mathbf{r}_1, \mathbf{r}_2, \cdots; t\right) = 0$$

e.g. (a rigid body) → $\left(\mathbf{r}_i - \mathbf{r}_j\right)^2 - c_{ij}^2 = 0$



non-holonomic examples: -Gas in a container
-Object rolling on a rough surface without slipping… more later

More quantifiers: - Rheonomous: depend on time explicitly
- Scleronomous: not explicitly depend on time

e.g. a bead constraints to move on a fixed vs. a moving wire

Constraints and Generalized Coordinates

Difficulties involving constraints:

1. Through $f(\mathbf{r}_1, \mathbf{r}_2, \cdots; t) = 0$, the individual coordinates $\boldsymbol{r}_i$ are *no* longer *independent*

   → eqs of motion (*) for individual particles are now *coupled* (not independent)

2. Forces of constraints are not known *a priori* and must be solved as additional unknowns

With **holonomic** constraints:

Prob #1 can be handled by introducing a set of "proper" (independent) Generalized Coordinates

Prob #2 can be treated with: D'Alembert's Principle & Lagrange's Equations (with Lagrange multipliers)

Generalized Coordinates

- Without constraints, a system of N particles has 3N dof
- With K constraint equations, the # dof reduces to 3N-K

- With holonomic constraints, one can introduce (3N-K) **independent** (proper) generalized coordinates $\left(q_1, q_2, \cdots, q_{3N-K}\right)$ such that:

$$\left.\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_1\left(q_1, q_2, \cdots, q_{3N-K}, t\right) \\ &\vdots \\ \mathbf{r}_N &= \mathbf{r}_N\left(q_1, q_2, \cdots, q_{3N-K}, t\right) \end{aligned}\right\}$$

$\mathbf{r}_i$ and $\left(q_1, \cdots q_{3N-K}\right)$ are related by a point transformation

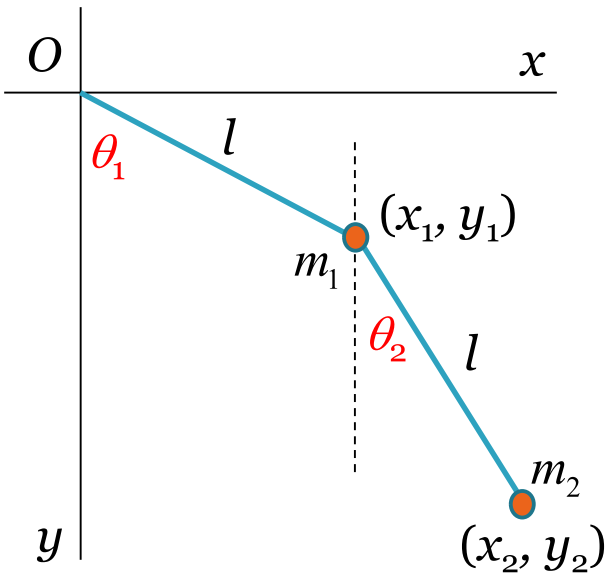
- The goal is to describe the time evolution of the system in the set of (3N-K) **independent** (proper) generalized coordinates.

Generalized Coordinates

- Generalized coordinates can be anything: angles, energy units, momentum units, or even amplitudes in the Fourier expansion of $\mathbf{r}_i$
- Each q_j is just a number, a scalar
- But, they must completely specify the state of a given system
- The choice of a particular set of generalized coordinates is not unique.
- No specific rule in finding the most "suitable" (resulting in simplest EOM)

Generalized Coordinates

Example:



(Double Plane Pendulum)

In Cartesian coord $\{\mathbf{r}_i\}$:

(x_1, y_1, x_2, y_2) we have 4 dof

2 constraints: $$\begin{cases} x_1^2 + y_1^2 - l^2 = 0 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - l^2 = 0 \end{cases}$$

So, there are only 2 indep dof…

One choice of generalized coords $\{q_j\}$ is:

(θ_1, θ_2) 2 *indep* dof

And, they are linked to the Cart. coord through:

(constraints are implicitly encoded in the Pt Trans)

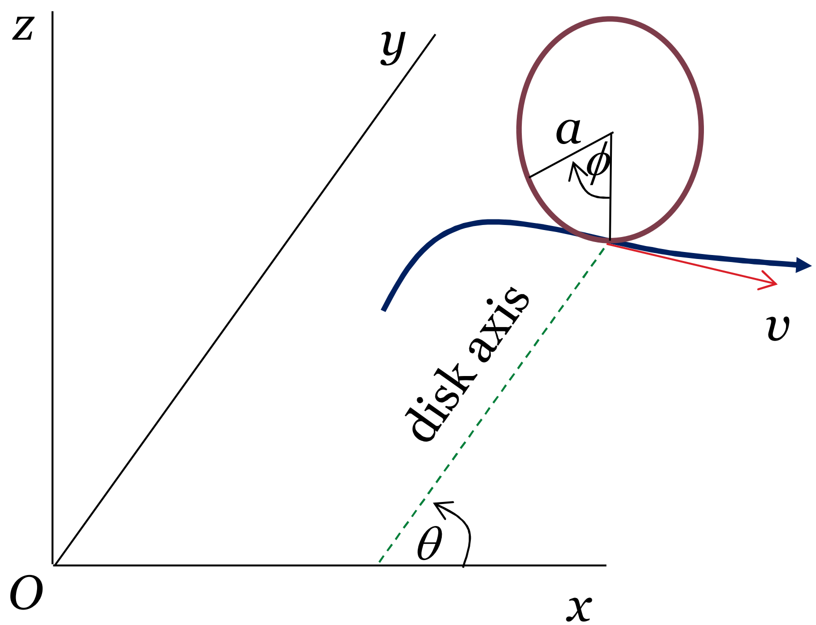
$$\theta_1 = \tan^{-1}(x_1 / y_1)$$

$$\theta_2 = \tan^{-1}\left((x_2 - x_1)/(y_2 - y_1)\right)$$

Non-Holonomic Constraints

- can't use constraint equations to eliminate dependent coordinates
- in general, solution is problem specific.

Example in book: Vertical Disk rolling without slipping on a horizontal plane



Described by 4 coordinates:

(x, y) of the contact point

θ : orientation of disk-

angle of disk axis with x-axis

ϕ : angle of rotation of the disk

Non-Holonomic Constraints (rolling disk exp)

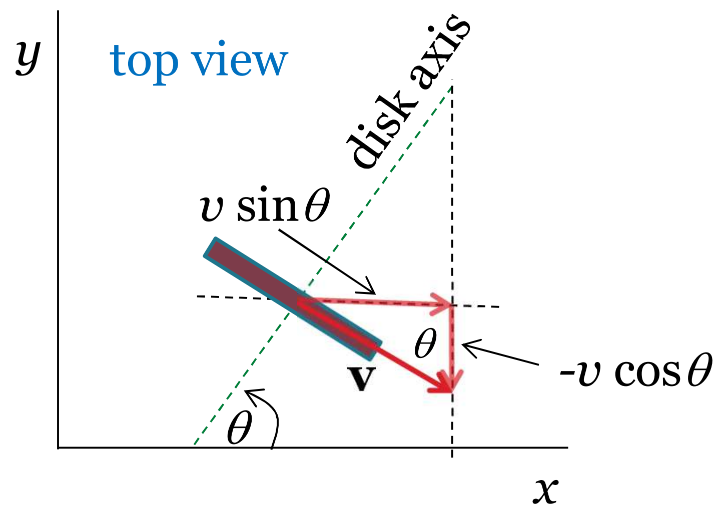
Now, consider the constraints:

1. No-slip condition:

$$s = a\phi \quad \Rightarrow \quad v = a\dot{\phi}$$

2. Disk rolling vertically

$\rightarrow$ $\mathbf{v} \perp$ disk axis $\xrightarrow{\text{see graph}}$



$$\Rightarrow \begin{cases} \dot{x} = v\sin\theta \\ \dot{y} = -v\cos\theta \end{cases}$$

Non-Holonomic Constraints (rolling disk exp)

Putting them together, gives the following *differential equations* of constraint,

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = a\dot{\phi}\sin\theta = a\sin\theta\dfrac{d\phi}{dt} \\ \dfrac{dy}{dt} = -a\dot{\phi}\cos\theta = -a\cos\theta\dfrac{d\phi}{dt} \end{cases} \quad or \quad \begin{cases} dx - a\sin\theta\, d\phi = 0 \\ dy + a\cos\theta\, d\phi = 0 \end{cases}$$

The point is that we can't write this in Holonomic form:

$\phi - f\left(x, y, \theta\right) = 0$ with f being a function! (hw)

Physical intuition → Roll the disk in a circle with radius R.

Upon completion of the circle, x, y and θ will have returned to their original values → but, ϕ will depend on R (can't be specified by $\left(x, y, \theta\right)$)

How to deal with Constraints?
Principle of Virtual Work

Consider the simplest situation, a system in *equilibrium* first,

- The net force on each particle vanishes: $\mathbf{F}_i = 0$ (note: *i* labels the particles)

Consider an arbitrary "virtual" infinitesimal change in the coordinates, $\delta \mathbf{r}_i$

- Virtual means that it is done with *no change in time* during which forces and constraints do *not* change.
- These virtual displacements are done *consistent* with the constraints (we will be more specific later).

Since the net force on each particle, $\mathbf{F}_i$ is zero (equilibrium), obviously we have:

$$\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0$$

↑ (virtual work)

Principle of Virtual Work

Separating the forces into applied $\mathbf{F}_i^{(a)}$ and constraint forces $\mathbf{f}_i$,

$$\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i$$

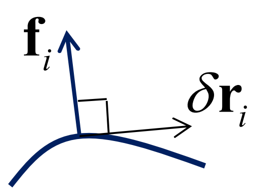
Then, we have $\sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$

Now, what do we mean by the virtual displacements $\delta \mathbf{r}_i$ being done *consistent* with the constraint?

Principle of Virtual Work

For virtual displacements to be *consistent* with the constraints means that

→ the *virtual work done by the constraint forces along the virtual displacement must be zero*.



$$\mathbf{f}_i \perp \delta\mathbf{r}_i \quad \text{or} \quad \mathbf{f}_i \cdot \delta\mathbf{r}_i = 0$$

Geometric view

>> More on this later for
N particles with K constraints, motion is restricted on a (3N-K)-D surface and the constraint forces $\mathbf{f}_i$ must be ⊥ to that surface.

For $\delta\mathbf{r}_i$ to be consistent with constraints means that the net virtual work

from the forces of constraints is zero !

$$\sum_i \mathbf{f}_i \cdot \delta\mathbf{r}_i = 0$$

Principle of Virtual Work

Back to our original equation for a constrained system in equilibrium,

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i + \underbrace{\sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i}_{0} = 0$$

With the virtual displacements satisfying the constraints leaves us with the statement,

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i = 0$$

→ The virtual work of the applied forces must also vanish!

This is called the Principle of Virtual Work.

Principle of Virtual Work

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i = 0$$

Note: Since the coordinates (and the virtual variations) are not necessarily independent. They are linked through the constraint equations. The Principle of Virtual Work does **not** imply,

$$\mathbf{F}_i^{(a)} = 0 \quad \text{for all } i \text{ independently.}$$

The trick is now to change variables to a set of *proper* (*independent)* generalized coordinates. Then, we can rewrite the equation as,

$$\sum_j (?)_j \cdot \delta q_j = 0$$

With q_j being independent, we can then claim: $(?)_j = 0$ for all j. As we will see, this will give us expressions which will lead to the solution of the problem.

D'Alembert's Principle

Now, we consider the more general case when the system is not necessary in equilibrium so that the net force on the particles is NOT zero. We continue to assume the constraints forces to be unknown *a priori*…

Similar to our discussion on the Principle of Virtual Work, we would like to reformulate the mechanical problem to include the constraint forces such that they "disappear" → you solve the "new" problem using only the (given) applied forces.

This is the basis for the D'Alembert's Principle AND by additionally choosing a set of proper **generalized coordinates**, the problem can be solved and it will result in the Euler-Lagrange's Equations.

D'Alembert's Principle

In deriving the Principle of Virtual Work, the system was in equilibrium.

In extending it to include dynamics , we will begin with Newton's 2nd law,

$$\mathbf{F}_i = \dot{\mathbf{p}}_i \quad \text{or} \quad \mathbf{F}_i - \dot{\mathbf{p}}_i = 0$$ for the i^{th} particle in the system.

We again consider a virtual infinitesimal displacement $\delta \mathbf{r}_i$ consistent with the given constraint. Since we have $\mathbf{F}_i - \dot{\mathbf{p}}_i = 0$ for all particles,

We have,

$$\sum_i \left(\mathbf{F}_i - \dot{\mathbf{p}}_i \right) \cdot \delta \mathbf{r}_i = 0$$

Again, we separate out the applied and constraint forces, $\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i$

This gives,

$$\sum_i \left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i \right) \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$$

D'Alembert's Principle

Then, following a similar argument for the virtual displacement to be consistent with constraints, i.e, $\sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$ (no virtual work for $\mathbf{f}_i$)

We can write down, $\sum_i \left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i \right) \cdot \delta \mathbf{r}_i = 0$

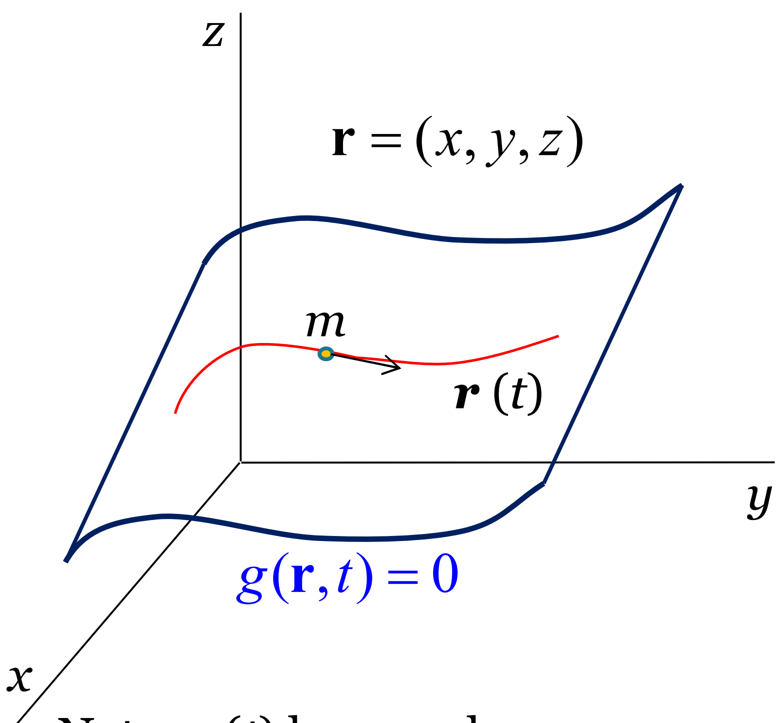
This is the D'Alembert's Principle.

Again, since the coordinates (and the virtual variations) are not necessary independent. This does NOT implies, $\left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i \right) = 0$ for the individual *i*.

We then need to look into changing variables to a set of *independent* generalized coordinates so that we have $\sum_j (?)_j \cdot \delta q_j = 0$ with the coefficients in the sum independently equal to zero, i.e., $(?)_j = 0$

Geometric View of the D'Alembert's Principle

Consider a particle moving in 3D with one Holonomic constraint,



$$\begin{cases} \text{equation of motion: } & m\ddot{\mathbf{r}} = \mathbf{F}^{(a)} + \mathbf{f} \\ \text{equation of constraint: } & g(\mathbf{r}, t) = 0 \end{cases}$$

Here,

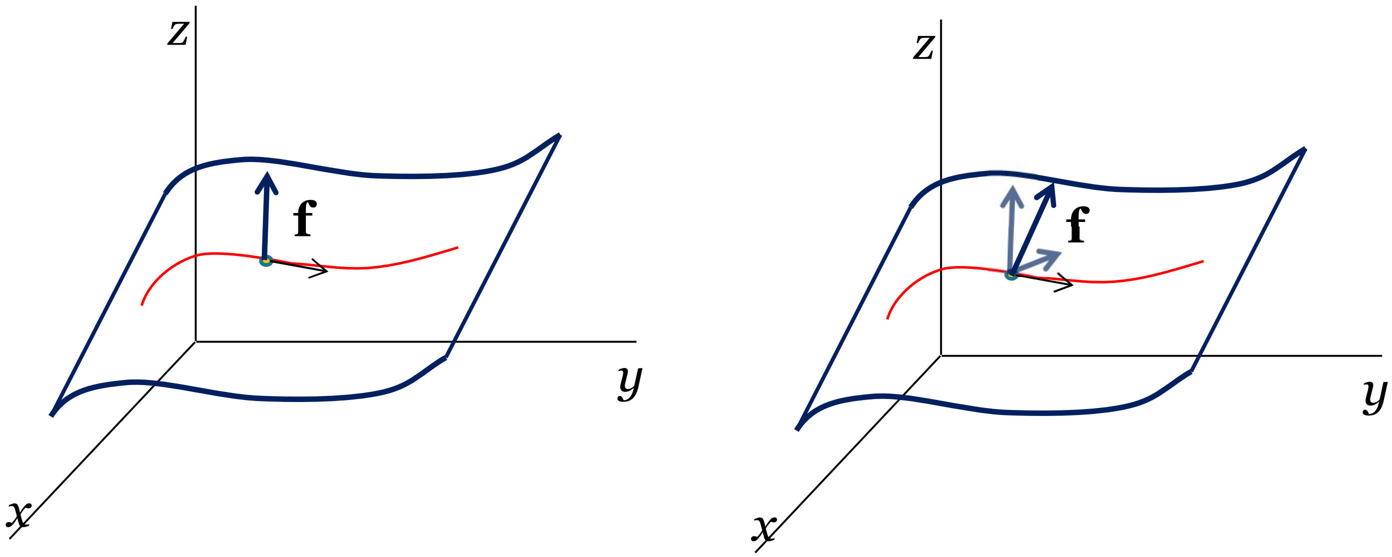
- $\mathbf{F}^{(a)}$ is the known applied force
- And, we model the unknown constraint force by the vector **f**.

Note: $\mathbf{r}(t)$ has 3 unknown components + 1 constraint ⇒ trajectory (red) is constraint to move in a 3-1=2 dimensional surface (blue $g(\mathbf{r}, t) = 0$).

Geometric View of the D'Alembert's Principle

- There are three unknown components to the constraint force **f**. A scalar constraint does not specify the vector **f** completely.

- There are multiple choices for **f** which satisfy $g(\mathbf{r}, t)=0$ BUT there is an additional physical restriction on **f** that we should consider…

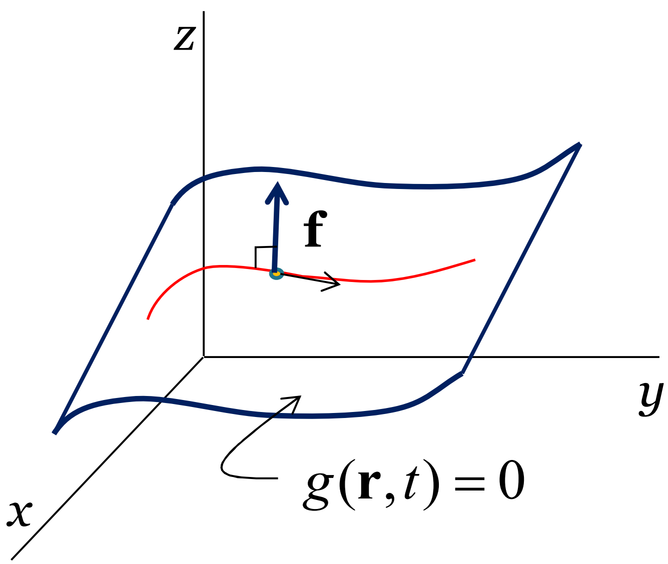


Observation: For a given **f**, adding a component // to the surface will still keep the particle **on** the surface (satisfying $g(\mathbf{r}, t)=0$) **but** will result with an *additional* acceleration along the surface).

Geometric View of the D'Alembert's Principle

→ A reasonable physical argument is to restrict the choice of **f** so that:

Constraint Force **f** needs to lay $\perp$ to the constraint surface



Note that $g(\mathbf{r},t)=0$ is the equation for the constraint surface and

$$\rightarrow \nabla g(\mathbf{r},t) \quad \perp \quad \text{surface}$$

So, we can "parametrized" **f** in term of $g(\mathbf{r},t)$,

$$\mathbf{f} = \lambda \nabla g(\mathbf{r},t) \qquad \text{where } \lambda \text{ is a parameter}$$

This gives,

$$\left.\begin{aligned} m\ddot{\mathbf{x}} &= \mathbf{F}^{(a)} + \lambda \nabla g(\mathbf{r},t) \\ g(\mathbf{r},t) &= 0 \end{aligned}\right\} \begin{array}{l} \text{4 unknowns } \mathbf{r} \text{ and } \lambda \\ \text{4 equations} \end{array}$$

Geometric View of the D'Alembert's Principle

$$\left.\begin{aligned} m\ddot{\mathbf{r}} &= \mathbf{F}^{(a)} + \lambda \nabla g(\mathbf{r},t) \\ g(\mathbf{r},t) &= 0 \end{aligned}\right\}$$

4 unknowns $\mathbf{r}$ and λ
4 equations

This system is solvable but now we would like to solve the system w/o using the constraint explicitly …

Note that ∇g is $\perp$ to the surface of constraint and we can project the dynamical equation onto the tangent plane of the constraint surface at ($\mathbf{r}$, t).

To do that, take $\mathbf{e}_a$ and $\mathbf{e}_b$ as two basis vectors spanning the tangent plane to the constraint surface at ($\mathbf{r}$, t). Dotting the above Eq to $\mathbf{e}_a$ and $\mathbf{e}_b$ gives two independent scalar equations,

$$\left(m\ddot{\mathbf{r}} - \mathbf{F}^{(a)}\right)\cdot \mathbf{e}_a = \lambda \nabla g(\mathbf{r},t)\cdot \mathbf{e}_a = 0$$

$$\left(m\ddot{\mathbf{r}} - \mathbf{F}^{(a)}\right)\cdot \mathbf{e}_b = \lambda \nabla g(\mathbf{r},t)\cdot \mathbf{e}_b = 0$$

Geometric View of the D'Alembert's Principle

Together with the constraint equation itself, we then have 3 eqs for the 3 unknown components of **r**.

$$\left.\begin{aligned} \left(m\ddot{\mathbf{r}} - \mathbf{F}^{(a)}\right)\cdot \mathbf{e}_{a,b} &= 0 \\ g(\mathbf{r}, t) &= 0 \end{aligned}\right\}$$

3 unknowns **r**
3 equations

So, now, in principle, we can solve for the dynamical equation (EOM), **r**(t), without knowing the constraint forces **f** explicitly.

$$\left(m\ddot{\mathbf{r}} - \mathbf{F}^{(a)}\right)\cdot \mathbf{e}_{a,b} = 0$$

→ This is the D'Alembert's Principle (for a single particle).

Geometric View of the D'Alembert's Principle

We can generalize the argument to a system of *N* particles with *K* constraints (Holonomic):

$$\sum_i \left(m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i^{(a)} \right) \cdot \mathbf{e}_k = 0$$

$$\left(\sum_i \left(\dot{\mathbf{p}}_i - \mathbf{F}_i^{(a)} \right) \cdot \delta \mathbf{r}_i = 0 \right)$$

Note: The virtual $\delta \mathbf{r}_i$ displacements consistent with the constraints are in the tangent space spanned by the basis $\{\mathbf{e}_k\}$

Geometric Interpretation:

The *K* constraints restrict the system to a (3*N*-*K*)-D surface within the 3*N*-D space. There are (3*N*-*K*) $\mathbf{e}_k$ vectors spanning that tangent plane to the constraint surface so that the above expression gives (3*N*-*K*) equations that the problem can be solved without knowing the constraint forces explicitly.

D'Alembert's Principle

$$\sum_i \left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i\right) \cdot \delta \mathbf{r}_i = 0$$

To solve for EOM using the D'Alembert's Principle …

We still need to look into changing variables to a set of *independent* **generalized coordinates** so that we have

$$\sum_j (?)_j \cdot \delta q_j = 0$$

Then, we can claim the "coefficients" $(?)_j$ in the sum to be independently equal to zero, i.e.,

$$(?)_j = 0$$

D'Alembert's Principle

$$\sum_i \left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i\right) \cdot \delta \mathbf{r}_i = 0$$

To solve for EOM using the D'Alembert's Principle ...

We still need to look into changing variables to a set of *independent* **generalized coordinates** so that we have

$$\sum_j (?)_j \cdot \delta q_j = 0$$

The correct "coefficients" allowing us to have $(?)_j = 0$ will give us the **Euler-Lagrange equation** and the EL Eq gives an explicit expression for the EOM:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = 0$$

Side Note: Constraint and Work

Recall that we have from the EOM: $m\ddot{\mathbf{r}} = \mathbf{F}^{(a)} + \lambda \nabla g(\mathbf{r},t)$

Let $\mathbf{F}^{(a)}$ be a conservative force , i.e., $\mathbf{F}^{(a)} = -\nabla U(\mathbf{r},t)$ so that

$$m\ddot{\mathbf{r}} = -\nabla U + \lambda \nabla g$$

Dotting $\dot{\mathbf{r}}$ into both sides,

$$m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}} = \frac{d}{dt}\left(\frac{1}{2}m\dot{\mathbf{r}}^2\right) = \frac{dT}{dt} \qquad\qquad -\nabla U\cdot\dot{\mathbf{r}} + \lambda\nabla g\cdot\dot{\mathbf{r}}$$

Consider the full-time derivative of g , we have,

$$\frac{dg}{dt} = \left(\frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt}\right) + \frac{\partial g}{\partial t} = \left(\nabla g\cdot\dot{\mathbf{r}}\right) + \frac{\partial g}{\partial t}$$

An Aside: Constraint and Work

As the particle moves, it is constraint to stay on the g=0 surface,

So, $\frac{dg}{dt} = 0$ and, $\left(\nabla g \cdot \dot{\mathbf{r}}\right) = -\frac{\partial g}{\partial t}$

Similarly, considering the full-time derivate of U, $\nabla U \cdot \dot{\mathbf{r}} = \frac{dU}{dt} - \frac{\partial U}{\partial t}$

Putting everything together,

$$m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = -\nabla U \cdot \dot{\mathbf{r}} + \lambda \nabla g \cdot \dot{\mathbf{r}}$$

$$\frac{dT}{dt} = -\frac{dU}{dt} + \frac{\partial U}{\partial t} - \lambda \frac{\partial g}{\partial t}$$

With $E=T+U$,

$$\frac{dE}{dt} = \frac{\partial U}{\partial t} - \lambda \frac{\partial g}{\partial t}$$

An Aside: Constraint and Work

$$\frac{dE}{dt} = \frac{\partial U}{\partial t} - \lambda \frac{\partial g}{\partial t}$$

So, either U or g *explicitly* depends on time, the total energy changes with time.

Since we typically do not consider time-dependent U potential functions,
So, we can make the following assertions:

Scleronomous (g not explicitly depends on t) Holonomic Constraints:

$$\left(\nabla g \cdot \dot{\mathbf{r}}\right) = -\frac{\partial g}{\partial t} = 0$$ and constraint force won't do work!

Rheonomous (g explicitly depends on t) Holonomic Constraints:

$$\left(\nabla g \cdot \dot{\mathbf{r}}\right) = -\frac{\partial g}{\partial t} \neq 0$$ and constraint force can do work!